Some more intrusion into the Dirac eqn

Unanswered Questions : How does the mass change sign?
↳ we'll see a recipe today

$$H = v\vec{p}\cdot\vec{\alpha} + \underbrace{(mv^2 - Bp^2)}_{\text{like a spin changing term}}\beta \quad \begin{cases} mB>0 \\ mB<0 \end{cases}$$

⇒ eff. hamiltonian for some lattice model

Sol^n in 1D (note that we won't assume a changing $m(x)$ here from -ve to +ve)

$$\mathcal{H} = v\, p_x \sigma_x + (mv^2 - Bp_x^2)\sigma_z$$

look for 0 en mode

$$H\psi = 0$$

$$\Rightarrow \left[v\, p_x\, \sigma_x + (mv^2 - Bp_x^2)\,\sigma_z\right]\psi = 0$$

$$\rightarrow \quad v\, p_x\, \psi = (mv^2 - Bp_x^2)\, -i\sigma_y)\psi$$

$$\Rightarrow \quad +v\, \partial_x\psi = (mv^2 + B\partial_x^2)\,\sigma_y\,\psi$$

Choose ψ s.t. $\psi = \varphi_\eta\, \phi(x)$ s.t. $\sigma_y \varphi_\eta = \eta\varphi_\eta$, $\eta = \pm 1$

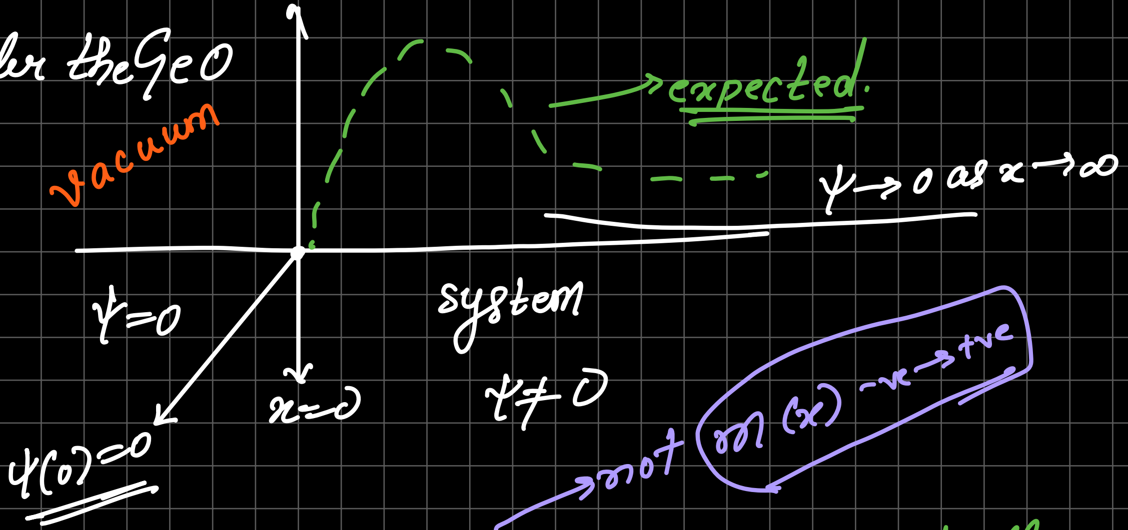
$$\Rightarrow -\frac{v}{\eta}\partial_x\phi = (mv^2 + B\partial_x^2)\phi(x) \xrightarrow{\phi = e^{tx}} -\frac{tv}{\eta} = mv^2 + Bt^2$$

$$\Rightarrow \quad Bt^2 + \frac{t}{\eta}v + mv^2 = 0$$

sum:- $\quad -t_+ - t_- = \dfrac{v}{\eta B} \qquad (-t_+)(-t_-) = \dfrac{mv^2}{B}$

$$\Rightarrow t = \frac{v}{2B}\left[\frac{-1}{\eta} \pm \sqrt{1 - 4Bm}\,\right] \begin{matrix} \nearrow -t_- \\ \searrow -t_+ \end{matrix} \Bigg\} \text{ for } t_-, t_+ < 0, \quad \eta = \mathrm{sgn}(B) \text{ \& } mB > 0$$

now consider the GeO



BC :- $\psi(0) = 0$ & $\psi(\infty) \to 0$

exponentially decaying wavefunction (in the bulk)

$$\therefore \psi = \varphi_\eta \left[a e^{-t_+ x} + b e^{-t_- x} \right]$$

clearly $t_\pm > 0$

$$@ x=0, \quad b = -a \quad \therefore \quad \psi = \varphi_\eta \left[e^{-t_+ x} - e^{-t_- x} \right] . N$$

$$t_- \sim \left(\frac{v}{2B}\right)\left[\frac{1}{\eta} - \sqrt{1 - 4mB}\right] \qquad t_+ \sim \frac{v}{2B}\left[\frac{1}{\eta} + \sqrt{1-4mB}\right]$$

Two length scales $\xi_+ = \frac{1}{t_+}$ & $\xi_- = \frac{1}{t_-}$ emerge.

Limits

① $B \to 0$, $\quad t_+ \to \infty$, $t_- \sim \frac{v}{2B}\left[1 - 1 + 2mB\right] \xrightarrow{\eta = 1} \frac{vm}{2} \sim$ const.

$\xi_+ \to 0$, $\qquad \xi_- = \frac{1}{mv}$

not an edge mode characteristic

while $E_{gap} = mv^2$ i.e. $\xi \propto \frac{1}{Gap}$

$\therefore$ we still get an edge mode provided $m \not\to \infty$

If $m \to \infty$, $\quad \xi_\pm \to \infty \implies$ Bulk state

$\therefore$ $B \longrightarrow 0 + m \longrightarrow 0$ is a topological phase transition

(Why are these called "Quantum Phase transition"?)

$$\therefore \psi_n(x) = \begin{pmatrix} 1 \\ i \cdot \mathrm{sgn}(B) \end{pmatrix} \left(e^{-x/\xi_+} - e^{-x/\xi_-} \right)$$

But to gen to 2D, we require 4 component form

$$H = v p_x \alpha_x + \underbrace{v p_y \alpha_y + v p_z \alpha_z}_{0} + (mv^2 - Bp^2)\beta$$

$$\therefore H = v p_x \alpha_x + (mv^2 - Bp^2)\beta$$

now from the note on Dirac eqn (see D. eqn recap)

we k.t. $E_\uparrow \neq E_\downarrow$ if $p_y \neq 0$

but here $p_y = 0 \Rightarrow E_\uparrow = E_\downarrow$

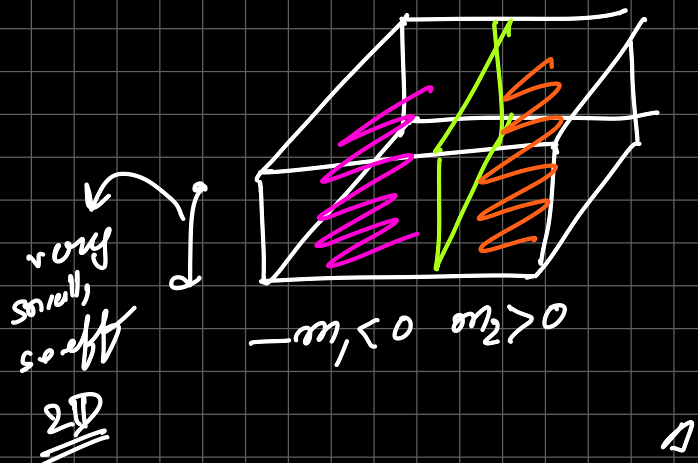
$\therefore$ 2 DEGENERATE Spinors are

$$\left.\begin{aligned} \psi_1 &= \frac{C}{\sqrt{2}} \begin{pmatrix} \mathrm{sgn}(B) \\ 0 \\ 0 \\ i \end{pmatrix} \uparrow \left(e^{-x/\xi_+} - e^{-x/\xi_-} \right) \\ \psi_2 &= \frac{C}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathrm{sgn}(B) \\ i \\ 0 \end{pmatrix} \downarrow \left(e^{-x/\xi_+} - e^{-x/\xi_-} \right) \end{aligned}\right\}$$ these can be leveraged to get higher D sol^{n}s

Two Dimensions Helical edge state

$$h_{\pm} = v\sigma_x p_x \pm v\sigma_y p_y + (mv^2 - Bp^2)\sigma_z$$

- $p_y = \hbar k_y$ is a good Q.No.



idea:- take 1D sol^n $\psi_{1,2}$ & treat $\Delta H = (\pm v\sigma_y p_y) - Bp_y^2\sigma_z$ as as a perturbation

$$\Delta H = v\,\alpha_y p_y - B p_y^2 \beta$$

$$H_{eff} = (h + \Delta H)_{in\ \psi_1, \psi_2} = \begin{pmatrix} \langle\psi_1|H_{eff}|\psi_1\rangle & \langle\psi_1|H_{eff}|\psi_2\rangle \\ \langle\psi_2|H_{eff}|\psi_1\rangle & \langle\psi_2|H_{eff}|\psi_2\rangle \end{pmatrix}$$

change $|\psi_1\rangle_{x,y} \to \underline{\psi_1 \times e^{ik_y y}}$

$$\langle\psi_1|\Delta H|\psi_1\rangle = (v k_y)\,(\mathrm{sgn}B \ \ 0 \ \ 0 \ \ -i)\begin{pmatrix}1\\0\\0\\ i\,\mathrm{sgn}B\end{pmatrix}$$

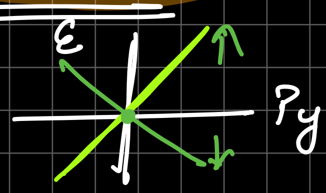
$$= \underline{\underline{2\,\mathrm{sgn}B\, v k_y}} + 0$$

Also, $\langle\psi_i|\Delta H|\psi_j\rangle = 0$ for $\underline{\underline{i \neq j}}$

$$\therefore \Delta H = [\langle\psi_1|, \langle\psi_2|]\ \Delta H \begin{pmatrix}|\psi_1\rangle\\|\psi_2\rangle\end{pmatrix} = \underline{\underline{v p_y\, \mathrm{sgn}(B)\, \sigma_z}}$$

$\therefore$ $\psi_\uparrow = \psi_1 : E(\psi_1) = +v p_y\, \mathrm{sgn}(B)$

$\psi_\downarrow = \psi_2 : E(\psi_2) = -v p_y\, \mathrm{sgn}(B)$



① $B = 0 \longrightarrow$ helical edge state disappear

② $\uparrow \longrightarrow$ +ve $v_g = \frac{\partial E}{\partial k}$ $\Big\}$ form helical edge states

$\downarrow \longrightarrow$ −ve v_g

Solving exactly,

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The exact solutions of the edge states in this two-dimensional equation have the form similar to that in the one-dimensional equation [5],

$$\Psi_1 = \frac{C}{\sqrt{2}} \begin{pmatrix} sgn(B) \\ 0 \\ 0 \\ i \end{pmatrix} (e^{-x/\xi_+} - e^{-x/\xi_-}) e^{+ip_y y/\hbar} \tag{2.48}$$

and

$$\Psi_2 = \frac{C}{\sqrt{2}} \begin{pmatrix} 0 \\ sgn(B) \\ i \\ 0 \end{pmatrix} (e^{-x/\xi_+} - e^{-x/\xi_-}) e^{+ip_y y/\hbar}, \tag{2.49}$$

with the dispersion relations $\epsilon_{p_y,\pm} = \pm v p_y \mathrm{sgn}(B)$. The characteristic lengths become p_y dependent,

$$\xi_\pm^{-1} = \frac{v}{2|B|\hbar}\left(1 \pm \sqrt{1 - 4mB + 4B^2 p_y^2/v^2}\right). \tag{2.50}$$

Chern number

For a $H = \vec{d}(\vec{p})\cdot\vec{\sigma}$ type H, C.no. is given by

$$n_c = -\frac{1}{4\pi}\int d\vec{p}\, \frac{\vec{d}\cdot(\partial_{p_x}\vec{d} \times \partial_{p_y}\vec{d})}{d^3}$$

here $n_c \longrightarrow$ can be fractional $\because$ $\vec{p}$ runs over $\mathbb{R}^2$ and not a BZ.

(over a BZ, it'll be an integer)

result:- $n_\pm = \pm\frac{1}{2}\left(\mathrm{sgn}(m) + \mathrm{sgn}(B)\right)$

$\Downarrow$

$mB > 0 \Rightarrow$ non-trivial topological system.

$\Downarrow$

"helical edge states"

(+ Gapped bulk)

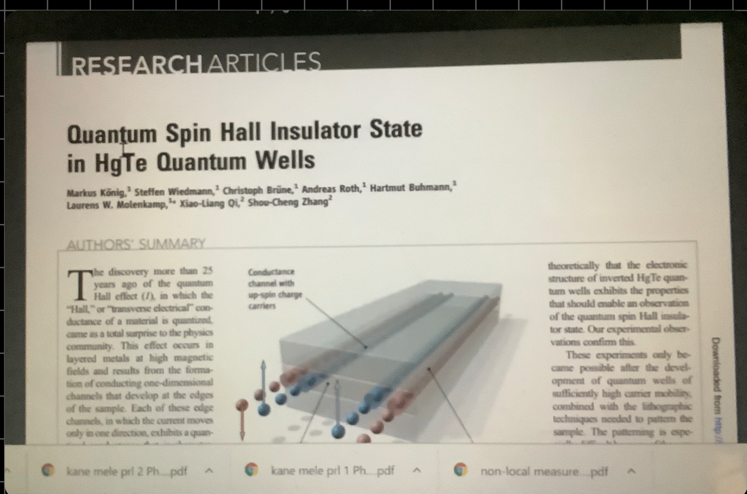
$mB < 0 \Rightarrow \eta = 0 \Rightarrow$ "no helical edge states"

Quite reminiscent of QSHall physics

??? : But we still don't know how to effect a $-m$ to m change?

↳ Do you even need it?

Expt details



RESEARCH ARTICLES

Quantum Spin Hall Insulator State in HgTe Quantum Wells

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AUTHORS' SUMMARY

The discovery more than 25 years ago of the quantum Hall effect (1), in which the "Hall," or "transverse electrical" conductance of a material is quantized, came as a total surprise to the physics community. This effect occurs in layered metals at high magnetic fields and results from the formation of conducting one-dimensional channels that develop at the edges of the sample. Each of these edge channels, in which the current moves only in one direction, exhibits a quan-

theoretically that the electronic structure of inverted HgTe quantum wells exhibits the properties that should enable an observation of the quantum spin Hall insulator state. Our experimental observations confirm this.

These experiments only became possible after the development of quantum wells of sufficiently high carrier mobility, combined with the lithographic techniques needed to pattern the sample. The patterning is espe-

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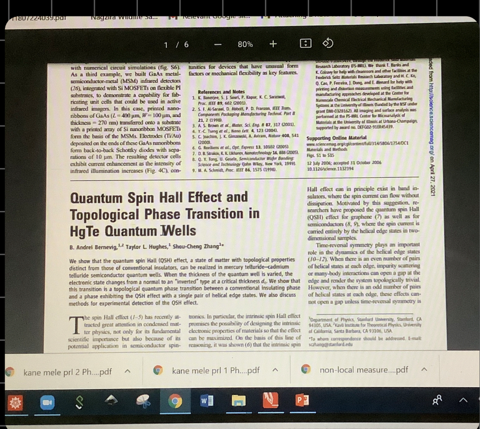
expt paper



Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells

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We show that the quantum spin Hall (QSH) effect, a state of matter with topological properties distinct from those of conventional insulators, can be realized in mercury telluride–cadmium telluride semiconductor quantum wells. When the thickness of the quantum well is varied, the electronic state changes from a normal to an "inverted" type at a critical thickness d_c. We show that this transition is a topological quantum phase transition between a conventional insulating phase and a phase exhibiting the QSH effect with a single pair of helical edge states. We also discuss methods for experimental detection of the QSH effect.

The spin Hall effect (1–5) has recently attracted great attention in condensed matter physics, not only for its fundamental scientific importance but also because of its potential application in semiconductor spintronics. In particular, the intrinsic spin Hall effect promises the possibility of designing the intrinsic electronic properties of materials so that the effect can be maximized. On the basis of this line of reasoning, it was shown (6) that the intrinsic spin Hall effect can in principle exist in band insulators, where the spin current can flow without dissipation. Motivated by this suggestion, researchers have proposed the quantum spin Hall (QSH) effect for graphene (7) as well as for semiconductors (8, 9), where the spin current is carried entirely by the helical edge states in two-dimensional samples.

Time-reversal symmetry plays an important role in the dynamics of the helical edge states (10–12). When there is an even number of pairs of helical states at each edge, impurity scattering or many-body interactions can open a gap at the edge and render the system topologically trivial. However, when there is an odd number of pairs of helical states at each edge, these effects cannot open a gap unless time-reversal symmetry is

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Theory proposal

Needs to be filled

(completed) $\Rightarrow$ after φ Hall

CdTe – HgTe
structure

Non-local transport

Summary:- H_{Dirac} has the property of

modified too

boundary modes anti-propagating

acc. to spin dof